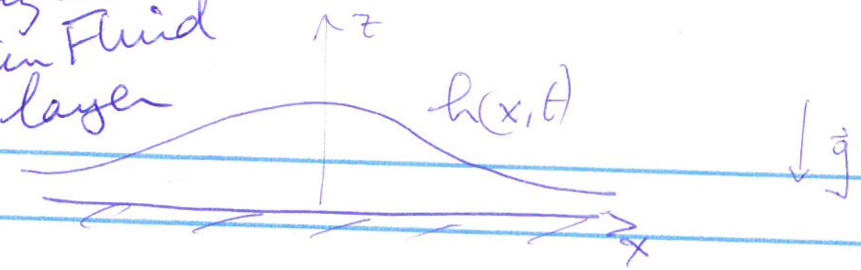
Sketchy notes on homework

for MA 451 Spring 2018

Horizontal Thin Fluid layer



$\alpha = 0$

x $\quad 0 = -\frac{1}{\rho}p_x + \nu u_{zz} + 0$

$0 = -\frac{1}{\rho}p_z - g$

$p = -\rho g(z-h) + \cancel{p_0(x,t)} \qquad p_x = +\rho g h_x$

$p_x = \mu u_{zz} \qquad u = \frac{g}{\nu} h_x\left(\frac{z^2}{2} - zh\right)$

w

$\cancel{w_z = -u_x = +\frac{g}{\nu} h_{xx} z h + \frac{g}{\nu} h_x^2 z - z^2}$

$h_t + u h_x = w = \frac{g}{\nu} h_{xx} h \frac{z^2}{2} - \frac{g}{\nu} h_x^2 \frac{z^2}{2}$ on $z=h$

$\cancel{h_t + \frac{g}{\nu}\frac{h^2}{2}h_x^2 = \frac{g}{\nu}\frac{h^3}{2}h_{xx} - \frac{g}{\nu}h_x^2\frac{h^2}{2}}$

$w_z = -u_x = -\frac{g}{\nu}\left\{h_{xx}\frac{z^2}{2} - z(h h_x)_x\right\}$

$w\Big|_{z=h} = -\frac{g}{\nu}\left\{h_{xx}\frac{h^3}{6} - \frac{h^2}{2} h h_{xx} - \frac{h^2}{2}h_x^2\right\}$

$\frac{1}{6} - \frac{1}{2} = \frac{1-3}{6} = -\frac{1}{3}$

on $z=h$ = kinematic BC

$h_t + u h_x = w : \quad h_t + \frac{g}{\nu}\left(-\frac{h^2}{2}\right)h_x^2 = -\frac{g}{\nu}\left(-\frac{h^3}{3}h_{xx} - \frac{h^2}{2}h_x^2\right)$

$h_t - \frac{g}{3\nu}h^3 h_{xx} - \frac{g}{\nu}h^2 h_x^2 = 0$

$h_t - \frac{g}{3\nu}\left(h^3 h_x\right)_x = 0$

Holmes Book 9.1–9.3 MA451

9.1 $u = ax^2 + bxy + cy^2$, (a) $v = -2axy - \frac{by^2}{2}$: $u_x + v_y = 0$.

$$\rho(\vec{v}_t + \vec{v}\cdot\nabla\vec{v}) = -\nabla p + \mu\Delta\vec{v}$$

(b) $T = -pI + 2\mu D = -pI + 2\mu\begin{pmatrix} 2a+by & \frac{1}{2}[bx+2(c-a)y] \\ \frac{1}{2}[bx+2(c-a)y] & -(2a+by) \end{pmatrix}$

$u_y + v_x = bx + 2cy - 2ay$
$= bx + 2(c-a)y$

(c) irrotational if $a=-c$, $b=0$.

9.2 $\vec{v} = (x+y, 3x-y, 0)$ (a) $\nabla\cdot\vec{v} = 0$

(b) $\vec{v}\cdot\nabla\vec{v} = \left((x+y)\partial_x + (3x-y)\partial_y\right)\begin{pmatrix} x+y \\ 3x-y \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} x+y+3x-y \\ 3x+3y-(3x-y) \\ 0 \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 0 \end{pmatrix}$$

(c) $= -\frac{1}{\rho}\nabla p + \mu\Delta\vec{v} \to 0$

$p_x = -4x \qquad p = -2x^2 + f(y)$

$p_y = -4y = f' \qquad f(y) = -2y^2$

$p = -2(x^2+y^2)$

(d) $\dot{x} = x+y \qquad \dot{y} = 3x-y \qquad x = ae^{\lambda t}$

$\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad y = be^{\lambda t}$

9.3(a),(e)

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)^2 = 3$$

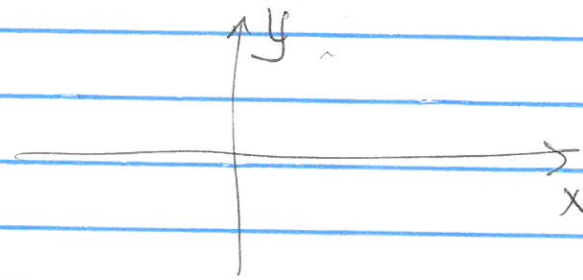
$$\lambda - 1 = \pm\sqrt{3}$$

$$\lambda = 1 \pm \sqrt{3}$$

$\lambda = 1+\sqrt{3}$: $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

$\lambda = 1-\sqrt{3}$ $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

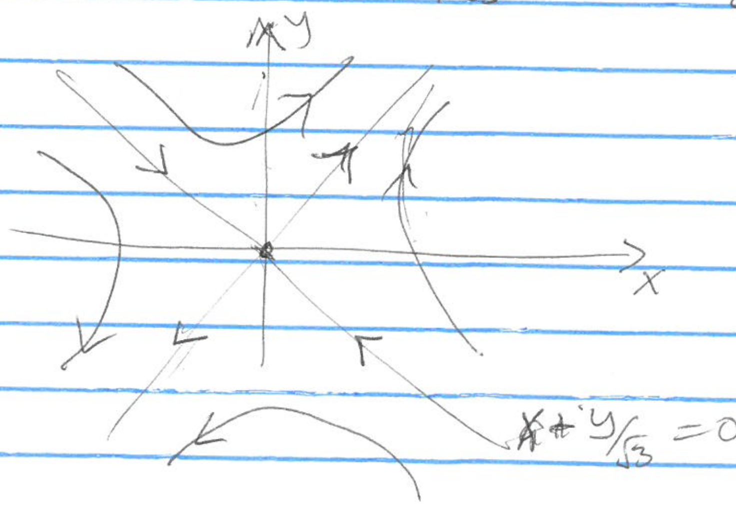
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a\, e^{(1+\sqrt{3})t} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} + b\, e^{(1-\sqrt{3})t} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$



$t=0$ $\quad a+b = x_0 \qquad a = \frac{1}{2}\left(x_0 + y_0/\sqrt{3}\right)$

$\quad a-b = y_0/\sqrt{3} \qquad b = \frac{1}{2}\left(x_0 - y_0/\sqrt{3}\right)$

(e)



$$\vec{v} = (x+y,\ 3x - y,\ 0)$$

9.3(f)

$$D = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}\, D = 0$ $\quad$ $\mathrm{II} = \det D = \lambda_1\lambda_2\lambda_3 = 0$

$\mathrm{III} = 0.$ $\quad$ $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$

$= \lambda_1\lambda_2 = -5.$

9.12 $\vec{v} = (-\alpha y, \alpha x, \beta)$

(a) $\nabla \cdot \vec{v} = 0 + 0 + 0.$

(b) $v \cdot \nabla v = -\nabla p + \mu \cancel{\Delta v}^{\,0.}$

$$(-\alpha y \partial_x + \alpha x \partial_y + \beta \partial_z)\begin{pmatrix} -\alpha y \\ \alpha x \\ \beta \end{pmatrix} = -\begin{pmatrix} \partial_x p \\ \partial_y p \\ \partial_z p \end{pmatrix}$$

$$\begin{pmatrix} -\alpha^2 x \\ -\alpha^2 y \\ 0 \end{pmatrix} = -\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \therefore \quad p = +\alpha^2 \frac{x^2}{2} + \alpha^2 \frac{y^2}{2} + p_0$$

$\nabla \times \vec{v} = u_y - v_x = -2\alpha$ rotational unless $\alpha = 0$

(c) pathlines $\dot{x} = -\alpha y \quad \dot{y} = \alpha x \quad \dot{z} = \beta \therefore z = \beta t + \cancel{A_3}\, z_0$

$$\ddot{x} = -\alpha \dot{y} = -\alpha^2 x$$

$x(t) = \cancel{\cos t}\, A_1 \cos \alpha t + \cancel{B_1} \sin \alpha t$

$y(t) = \cancel{\alpha}\, A_1 \sin \alpha t - B_1 \cos \alpha t \quad . \quad t = 0: \ -B_1 = A_2$

$$\begin{cases} x(t) = A_1 \cos \alpha t - A_2 \sin \alpha t \\ y(t) = A_1 \sin \alpha t + A_2 \cos \alpha t \\ z(t) = \beta t + A_3 \end{cases}$$

(d) rotation about z-axis with drift along axis, constant speed β.

helical flow.

9.5

$$u(y) = \begin{cases} A_1 y + B_1 & 0<y<h_0 \\ A_2 y + B_2 & h_0<y<h \end{cases}$$

$u(0) = 0 \Rightarrow B_1 = 0$

$p = p_0 = \text{const.}$

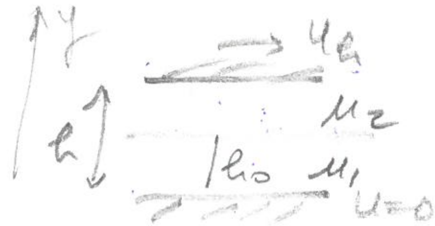
$h = h_0 \begin{cases} \tau|_{h=h_0} \text{ cont}^s \\ u \text{ conts} \end{cases}$

$\mu_1 A_1 = \mu_2 A_2$ (1)

$A_1 h_0 = A_2 h_0 + B_2$ (2)

$A_2 h + B_2 = u_h$ (3)

$u(h) = u_h$



density ρ_k plays no role.

(3) : $B_2 = u_h - A_2 h$

(2)(1) $A_1 h_0 = A_2 h_0 + u_h - A_2 h = \frac{\mu_2}{\mu_1} A_2 h_0$.

$$A_2\left(h - h_0 + \frac{\mu_2}{\mu_1} h_0\right) = u_h$$

$$A_2 = \frac{u_h \mu_1}{h_0 \mu_2 + \mu_1 (h - h_0)}$$

$$A_1 = \frac{u_h \mu_2}{h_0 \mu_2 + \mu_1 (h - h_0)}$$

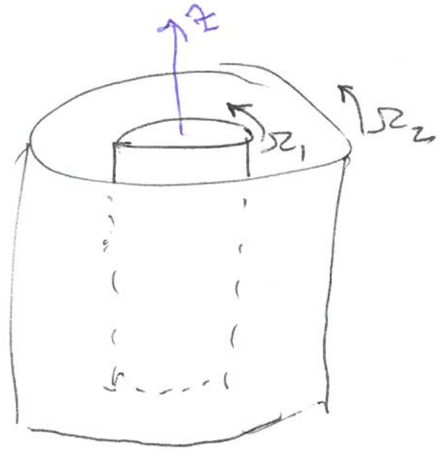
$B_2 = \cancel{u_h}$

$$u(y) = \begin{cases} A_1 y & y < h_0 \\ A_2 y + B_2 & y > h_0 \end{cases}$$

$$B_2 = u_h - A_2 h = u_h \left(1 - \frac{\mu_1 h}{h_0 \mu_2 + \mu_1 (h - h_0)}\right)$$

$$= \frac{h_0 (\mu_2 - \mu_1)}{h_0 \mu_2 + \mu_1 (h - h_0)} u_h$$

9.6



$\vec{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z$

$\hat{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$ $\hat{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$ $\hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\vec{u} = \cancel{\ } \; v(r)\,\hat{e}_\theta$$

$$v(r) = Ar + \frac{B}{r}$$

steady flow:

$$\vec{u}\cdot\nabla\vec{u} = -\frac{u_\theta^2}{r}\hat{e}_r = -\frac{v^2}{r}\hat{e}_r \qquad (v(r) = u_\theta)$$

$$\nabla^2\vec{u} = \left(\partial_r^2 + \frac{1}{r}\partial_r\right)(v\,\hat{e}_\theta) + \frac{1}{r^2}\partial_\theta^2(v\,\hat{e}_\theta)$$

$$= \left(\partial_r^2 v + \frac{1}{r}\partial_r v - \frac{v}{r^2}\right)\hat{e}_\theta$$

$$\therefore \quad -\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (\hat{e}_r)$$

$$0 = -\frac{1}{\rho r}\partial_\theta p + \nu\left(\partial_r^2 v + \frac{1}{r}\partial_r v - \frac{v}{r^2}\right)$$

$$0 = \partial_z p.$$

$$\therefore \quad p = p(r), \qquad \partial_r^2 v + \frac{1}{r}\partial_r v - \frac{v}{r^2} = 0$$

$$v = Ar + B/r.$$

$$v(r_1) = \Omega_1 r_1 \qquad v(r_2) = \Omega_2 r_2$$

$$\cancel{A_1 = \frac{\mu_2}{\mu_1} A_2}$$

$$\cancel{\therefore\ \mu_2\left(\lambda \frac{h_0^2}{2} + \mu_2 A_2 h_0\right) = \mu_1\left(\lambda \frac{h_0^2}{2} + \mu_2 A_2 h_0 + B_2\right)}$$

<u>9.6 cont^d.</u>

$$V = Ar + B/r.$$

$$Ar_1 + B/r_1 = \Omega_1 r_1 \qquad Ar_1^2 + B = \Omega_1 r_1^2$$

$$Ar_2^2 + B = \Omega_2 r_2^2$$

$$\therefore \quad B(r_2^2 - r_1^2) = (\Omega_1 - \Omega_2)\, r_1^2 r_2^2$$

$$A(r_2^2 - r_1^2) = \Omega_2 r_2^2 - \Omega_1 r_1^2$$

$$\partial_r P = \frac{\rho}{r} v^2 = \frac{\rho}{r}\left(Ar + \frac{B}{r}\right)^2$$

$$= \frac{\rho}{r}\left(A^2 r^2 + 2AB + \frac{B}{r^2}\right)$$

$$= \rho\left(A^2 r + \frac{2AB}{r} + \frac{B^2}{r^3}\right)$$

$$P = \rho\left(\frac{A^2}{2} r^2 + 2AB \ln r - \frac{B^2}{2r^2}\right) + p_0.$$
