MA 451 Methods of Applied Mathematics Homework Due March 29, 2018

Kinematic Boundary Condition at a Free Boundary:

A fluid layer flows over a flat solid horizontal surface. The upper surface of the fluid is free (like the surface of a river) and has height $z = h(x,t)$ where x, y are horizontal coordinates, t is time and the fluid flow is assumed independent of y. Each material point in the free surface, labeled by the Lagrangian variable A, stays in the surface as time progresses.

1. Consider the mapping $x = X(A,t), z = Z(A,t)$ relating Lagrangian to Eulerian coordinates to derive the kinematic boundary condition

$$w = h_t + uh_x,$$

at the free surface, where $\vec{v} = (u, 0, w)(x,t)$ is the velocity of the fluid.

2. Derive the corresponding kinematic boundary condition when the free surface $z = h(x,y,t)$ also depends on y.

Hint: draw a picture of material (Lagrangian) and physical (Eulerian) coordinates in each case, and the mappings between them.

3. Consider the spreading of a thin layer of viscous fluid on a horizontal rigid plate. Using the lubrication approximation $h/L << 1$ in the Navier-Stokes equations for incompressible flow, derive the equation for evolution of the free surface thickness (height) $z = h(x,t)$, assuming that the motion is independent of y :

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$