MA 493 First Test March, 2014

No electronics allowed, except to access the ebook

1. Consider a continuous one-dimensional bar initially located in the interval $0 \le x \le 1$. Suppose the motion of the bar is given in Lagrangian (material) coordinates by the deformation

$$X(A,t) = A(1 + a\sin t), \quad t \ge 0,$$

where a is a constant.

(a) Explain why $-1 < a < 1$ is needed.

Now let $\mathbf{a} = \frac{1}{2}$.

(b) Suppose at time $t = \frac{\pi}{2}$, a material point (a cross section of the bar) is located at $x = 1$, where was it at time $t = \frac{\pi}{4}$?

(c) Express the velocity in both Lagrangian and Eulerian (spatial) variables.

(d) Suppose the density of the bar, measured as mass per unit length, is $\rho = 5\,g/cm$ initially, and ρ is inversely proportional to the deformation gradient. What is the mass of the bar initially? Find the density of the bar at time $t = \frac{\pi}{4}$. (Hint: $\rho = \rho(t)$ depends on time but not position.)

(e) Does the assumption in part (d) make sense physically/intuitively? Explain.

(f) If the assumption in (d) is observed to be true, and suppose the density of the material (measured as mass per unit volume) is unchanged due to deformation. Does this mean the material is incompressible? Suppose the cross-section area of the bar initially is $4\,cm^2$. Assuming the deformation is uniform, find the cross-section area at time $t = \frac{\pi}{4}$.

2. Consider a fluid in three dimensions, and suppose the stress tensor T is given by the formula

$$\begin{bmatrix} x+y & z^2 & z \\ z^2 & y+z^2 & y \\ z & y & -z \end{bmatrix}.$$

(a) Calculate $\nabla \cdot T$.

(b) Let $\vec{v} = (u,v,w)(x,y,z,t)$ denote the velocity in Eulerian coordinates. Suppose there are no body forces, and the fluid is incompressible. Write the equations of motion in the variables u, v, w, x, y, z, t.

(c) Calculate the velocity if the fluid is initially at rest. You can take the density to be $\rho = 1$. Hint: the velocity depends only on time in Eulerian coordinates.

(d) Find the location $\vec{x} = (x,y,z)$ of a fluid particle at time $t = 2$ if initially it was at $(1,0,3)$.

(e) Explain why the velocity of part (c), when expressed in Lagrangian coordinates has the same formula as in Eulerian coordinates.