MA 451 First Test March, 2018

In problems 1 and 2, consider the longitudinal deformation and motion of an elastic rod modeled by one space variable x and time t. Let $x = X(A,t)$ denote the position at time t of a cross-section of the rod that is located at $x = A$ at time $t = 0$. Then $X(A,t)$ is the deformation at time t. All quantities are assumed to have been scaled, so that they are dimensionless.

1. Let $A = a(x,t)$ be the inverse of the deformation at time t.

(a) Write an equation that the function $a(x,t)$ must satisfy and hence show that

$$a_x = 1/X_A; \quad a_t = -X_t/X_A.$$

Answer: $x = X(a(x,t),t)$. Differentiate with respect to x and t to get the formulas.

$$(\partial/\partial x): \quad 1 = X_A a_x; \quad (\partial/\partial t): \quad 0 = X_A a_t + X_t.$$

(b) Write the velocity $V(A,t)$ in Lagrangian coordinates and the velocity $v(x,t)$ in Eulerian coordinates (using $a(x,t)$ and derivatives).

Answer: $V(A,t) = \partial_t X(A,t)$; $v(x,t) = \partial_t X(A = a(x,t),t) = -a_t X_A = -a_t/a_x$ from 1(a).

(c) Let $F(A,t) = X_A$ denote the deformation gradient in Lagrangian variable A and let $f(x,t)$ denote the deformation gradient in Eulerian coordinates. Show that

$$\frac{Df}{Dt} = \frac{\partial v}{\partial x} f,$$

where $\frac{D}{Dt}$ denotes the convective (or material) derivative. Hint: Express f in terms of $a(x,t)$ and use your expression for $v(x,t)$ from part (b).

Answer: Since $f = X_A = 1/a_x$, $v = -a_t/a_x$, we have $\dfrac{Df}{Dt} = f_t + v f_x = -\dfrac{a_{xt}}{a_x^2} + \dfrac{a_t}{a_x}\dfrac{1}{a_x^2}a_{xx}$. But from 1(b), $v_x = -\dfrac{a_{xt}}{a_x} + \dfrac{a_t}{a_x^2}a_{xx}$, so $\dfrac{Df}{Dt} = \dfrac{\partial v}{\partial x} f$.

2. Suppose the motion of a rod initially of length 2 is given by the specific deformation

$$X(A,t) = A(1+t) + A^2t^2.$$

(a) What is the length of the rod at time $t = 3$?

Answer: Let $A = 2, t = 3:$ $X(2,3) = 44$.

(b) Suppose a cross-section of the rod is located at $x = 1$ at time $t = 1$. Where will it be at time $t = 2$?

Answer: $X(A, 1) = 2A + A^2 = 1$ implies $A = -1 + \sqrt{2}$. The other root $A = -1 - \sqrt{2}$ is not physical, because it is not in the rod. Thus, at $t = 2$, $x = X(A, 2) = 3A + 4A^2 = 9 - 5\sqrt{2}$.

3. Consider a fluid in three dimensions, and suppose the stress tensor T is given by the formula

$$\begin{bmatrix} x+y & z^2+yt & z \\ z^2+yt & y+z^2 & y \\ z & y & -z \end{bmatrix}.$$

(a) Calculate $\nabla \cdot T$.

Answer: $\nabla \cdot T = \begin{bmatrix} 2+t \\ 1 \\ 0 \end{bmatrix}$.

(b) Let $\vec{v} = (u, v, w)(x, y, z, t)$ denote the velocity in Eulerian coordinates.
Suppose $\vec{v} = (xt, (x-y)t, 0)$. Show that such a motion satisfies the incompressibility condition.

Answer: $\nabla \cdot \vec{v} = t - t = 0$.

(c) Calculate $(\vec{v}.\nabla)\vec{v}$ and $\vec{v}_t$.

Answer: $(\vec{v}.\nabla)\vec{v} = \begin{bmatrix} xt^2 \\ yt^2 \\ 0 \end{bmatrix}$ and $\vec{v}_t = \begin{bmatrix} x \\ x-y \\ 0 \end{bmatrix}$.

(d) Write the Navier-Stokes equations incorporating the calculations of parts (a) and (c) to show that the motion of part (b) must be driven by body forces.

Answer:

$$\rho(\vec{v}_t + \vec{v}.\nabla\vec{v}) = \nabla \cdot T + \rho\vec{f}: \quad \rho \begin{bmatrix} x+xt^2 \\ x-y+yt^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+t \\ 1 \\ 0 \end{bmatrix} + \rho\vec{f}.$$

Therefore, $\vec{f} \neq 0$.