From Stakgold: Green's Functions and Boundary Value Problems

7. THE LEBESGUE INTEGRAL

No one will dispute the central role of Lebesgue integration in analysis, and this beautiful theory belongs in the repertoire of every aspiring mathematician. For our purposes, however, we shall need only the few facts listed at the end of the section and enough discussion to give them meaning. We can safely omit proofs because the methods involved are not used elsewhere in the book.

The Lebesgue integral may be viewed as an extension of the Riemann integral in the sense that every Riemann-integrable function is also Lebesgue integrable to the same value and that there exist some functions, such as the Dirichlet function described below, that fail to be Riemann integrable but are Lebesgue integrable. The kind of function that is Lebesgue integrable but not Riemann integrable rarely, if ever, occurs in practice. Thus it is not for computational reasons that the Lebesgue integral is so important. What the Lebesgue integral does is to give structural unity to analysis. From a philosophical point of view, the relationship of Lebesgue-integrable functions to Riemann-integrable functions is similar to that of real numbers to rational numbers. Concrete calculations require only rational numbers, but mathematics needs irrational numbers. The totality of real numbers (rational plus irrational) has an inner consistency absent from the class of rational numbers alone. It is the completeness (see Chapter 4) of the real number system which makes it powerful. Principally this means that when we apply limiting processes in the class of real numbers we remain within the class. Similarly we shall find that for most concrete calculations the notion of Riemann integral is adequate, but theorems involving passage to the limit are more easily formulated and proved within the class of Lebesgue-integrable functions.

The difference between these two concepts of integration is illustrated by the following analogy, which, though not strictly apt, has some anecdotal value. A shopkeeper can determine a day's total receipts either by adding the individual transactions (Riemann) or by sorting bills and coins according to their denomination and then adding the respective contributions (Lebesgue). Obviously the second approach is more efficient!

Consider now a nonnegative real-valued function $f(x)$ defined on the interval $0 \leqslant x \leqslant 1$. In the Riemann scheme one partitions the x interval, then forms the sum $\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$ for arbitrary ξ_k in $[x_{k-1}, x_k]$, and finally passes to the limit as $n \to \infty$ and the length of the largest subdivision tends to 0. The principal difficulty is proving that the limit exists independently of the choice of ξ_k. In the Lebesgue approach it is the y axis that is partitioned (see Figure 7.1). Let E_i be the set of values of x such that $y_{i-1} \leqslant f(x) < y_i$; in the favorable case shown in the figure, E_i is the union of a finite number of disjoint intervals. We then form the sum $\sum_{i=1}^{n} \eta_i m(E_i)$, where η_i is chosen arbitrarily in $[y_{i-1}, y_i]$ and $m(E_i)$ is the measure of E_i, that is, the sum of the lengths of the disjoint intervals that make up E_i. As the partition is made finer, there is no longer any question as to the existence of the limit of the sum. Indeed, the lower sum $\Sigma_{i=1}^{n} y_{i-1} m(E_i)$ is monotonically increasing with n and bounded above, and so must converge. The upper sum $\Sigma_{i=1}^{n} y_i m(E_i)$ differs from the lower sum by less than $\max(y_i - y_{i-1}) \Sigma_{i=1}^{n} m(E_i)$; since $\Sigma_{i=1}^{n} m(E_i) = 1$ and $\max(y_i - y_{i-1})$ tends to 0, the upper sum must also converge to the same value as the lower sum. This common value is the Lebesgue integral

$$\int_0^1 f(x)\,dx.$$

In less favorable cases the set E_i may be much more complicated than shown in the figure. A consistent definition for the measure of sets must be given so that the analysis just presented can be suitably adapted.

Rather than developing the measure-oriented approach to the Lebesgue integral, we prefer to follow the method of Tonelli for constructing the Lebesgue integral of a nonnegative real-valued function $f(x)$ on the interval $[0, 1]$. A set E of points on this interval is said to have *measure less than* ε if E can be contained in a finite or countably infinite set of intervals of total length less than ε. The set has *measure 0* if such a covering can be found for each $\varepsilon > 0$. A function $f(x)$ is *measurable* if, for any $\varepsilon > 0$, we can convert it into a continuous function by changing its values on a set of

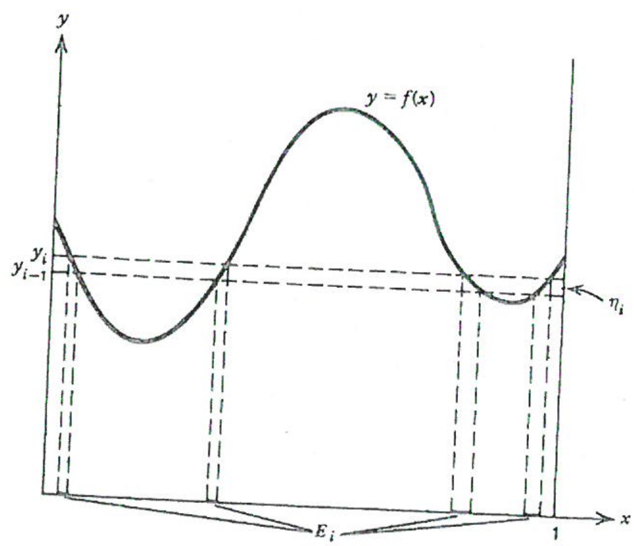

Figure 7.1

measure less than ε. Note that f is surely measurable if it can be converted to a continuous function by altering its values on a set of measure 0.

Example 1. Suppose $f(x)$ is piecewise continuous in $[0,1]$ with simple jumps at $x_1,\ldots,x_n$. In each of the intervals $[x_i-\varepsilon/2n, x_i+\varepsilon/2n]$ we can replace $f(x)$ by a straight line joining the points $(x_i-\varepsilon/2n, f(x_i-\varepsilon/2n))$ and $(x_i+\varepsilon/2n, f(x_i+\varepsilon/2n))$. The resulting function is continuous, and we have altered the values of the original function over a set of measure $n(\varepsilon/n)=\varepsilon$.

Example 2. The Dirichlet function $f(x)$ has the value 1 when x is rational and 0 when x is irrational. If we change the value of f from 1 to 0 on the set of rationals, we obtain the continuous function that vanishes identically on $[0,1]$. We claim the set of rationals has measure 0. Since the rational numbers form a countable set, they can be placed into 1-1 correspondence with the positive integers, for instance by ordering them so that they have increasing denominators:

$$0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \ldots.$$

For each $\varepsilon>0$, we enclose the kth rational number in this list in an interval of length $\varepsilon/2^k$. The total length of the intervals enclosing all rationals is then

$$\sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Thus for each $\varepsilon>0$ we can enclose the rational numbers in a countably infinite set of intervals whose total length is ε. Since this can be done for each $\varepsilon>0$, we conclude that the set of rationals has measure 0.

Therefore the *Dirichlet function is measurable.*

Example 3. Let $f(x)=x^{-\alpha}, 0<x\leqslant 1$, where $\alpha>0$; we define $f(0)$ to be 0, but any other value would do as well. Then $f(x)$ is measurable, since it can be converted into a continuous function by replacing it on $0\leqslant x<\varepsilon$ by the constant function $\varepsilon^{-\alpha}$. Of course there are many other ways in which such a conversion can be accomplished.

The notion of the Lebesgue integral of a nonnegative measurable function $f(x)$ can now be introduced. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n\to 0$. For each n construct a nonnegative continuous function $f_n(x)$ which differs from $f(x)$ only on a set of measure less than ε_n. Now $f_n(x)$ is certainly Riemann integrable on $0\leqslant x\leqslant 1$. Let us suppose that the functions $f_n(x)$ can be constructed so that *their integrals in the Riemann sense have a common bound*. Then the functions $f_n(x)$ can always be chosen so that their integrals form a convergent sequence. Since there are many possible choices for the sequence $\{f_n\}$, the limit of the sequence of integrals is not uniquely defined by $f(x)$ but can depend on the choice of the sequence $\{f_n\}$. The greatest lower bound of the possible limits is defined as the *Lebesgue integral* of $f(x)$:

$$\int_0^1 f(x)\,dx.$$

It is easily shown that every Riemann-integrable function is Lebesgue integrable to the same value. If $f(x)$ has an improper Riemann integral, as in Example 3 with $0<\alpha<1$, then $f(x)$ is Lebesgue integrable and the values of the integrals again coincide; if we take $\alpha \geqslant 1$ in Example 3, then neither the improper Riemann integral *nor* the Lebesgue integral exists. Thus the Lebesgue approach does not miraculously reduce infinite areas to finite values. However, the Dirichlet function of Example 2 is Lebesgue integrable to the value 0 but is not Riemann integrable (for any partition each subdivision contains both rational and irrational numbers, so that the Riemann sum can be made either 0 or 1 by choice of ξ_k).

If $f(x)$ can take both positive and negative values, we write $f(x)=f_+(x)-f_-(x)$, where f_+ and f_- are both nonnegative functions defined by

$$f_+=\begin{cases} f, & x\in E_+,\\ 0, & \text{elsewhere,}\end{cases}\qquad f=\begin{cases} -f, & x\in E_-,\\ 0, & \text{elsewhere,}\end{cases}$$

where E_+ and E_- are the sets on the x axis where f is positive and negative, respectively. It is then possible to define

$$\int_0^1 f\,dx=\int_0^1 f_+\,dx-\int_0^1 f_-\,dx.$$

The Lebesgue integral can also be defined for arbitrary finite intervals or for infinite intervals. The integral has the usual linearity property

$$\int_a^b [\alpha f(x)+\beta g(x)]\,dx=\alpha\int_a^b f(x)\,dx+\beta\int_a^b g(x)\,dx.$$

We have already remarked that the importance of the Lebesgue integral lies in the relative impunity with which we can use limiting processes in connection with Lebesgue integration. One of the most important theorems is the following.

Lebesgue Dominated Convergence Theorem. Let $\{s_n(x)\}$ be a sequence of integrable functions over $[a,b]$, which approaches a limit $s(x)$ pointwise except possibly over a set of measure 0. If there exists an integrable function $f(x)$ such that, for all sufficiently large n, $|s_n(x)| \leqslant f(x)$, then $s(x)$ is integrable and

$$\lim_{n\to\infty} \int_a^b s_n(x)\,dx = \int_a^b s(x)\,dx.$$

Note that this theorem is much more powerful than Theorem 2 based on uniform convergence in Section 6. Here we need only pointwise convergence (and then only almost everywhere), the interval does not need to be bounded, and the integrability of the limit is guaranteed by the theorem instead of having to be hypothesized.

Let us apply the theorem to sequence (6.2). If $\alpha < e$, we can show from elementary calculus that $\alpha \log y < y$ for all $y > 0$. Setting $y = nx$, we find $n^\alpha e^{-nx} < x^{-\alpha}$ or

$$s_n(x) < x^{1-\alpha}, \qquad x > 0. \tag{7.1}$$

Clearly $x^{1-\alpha}$ is integrable from 0 to 1 if $\alpha < 2$; since $2 \leqslant e$, (7.1) also holds and therefore the Lebesgue theorem yields (6.5) for $\alpha < 2$. In fact, we can refine (7.1) to show that, for $\alpha < 2$, $s_n(x) < f(x)$, $0 < x < \infty$, where $\int_0^\infty f(x)\,dx$ is finite. The Lebesgue theorem then tells us that

$$\lim_{n\to\infty} \int_0^\infty s_n(x)\,dx = 0, \qquad \alpha < 2.$$

The other principal fact we need to know about Lebesgue integration is the completeness of $L_2(a,b)$, the space of real-valued, square integrable functions on $a \leqslant x \leqslant b$. Thus $u(x) \in L_2(a,b)$ if and only if

$$\int_a^b u^2(x)\,dx < \infty.$$

In L_2 we use the distance function (6.8). The significance of the completeness of L_2 will be better appreciated on reading Chapter 4.

REFERENCES AND ADDITIONAL READING

Antman, S., General solutions for plane extensible elasticae having nonlinear stress-strain laws, *Quart. Appl. Math.* **26** (1968), 35.

Aris, R., *The mathematical theory of diffusion and reaction in permeable catalysts*, Vols. 1 and 2, Oxford University Press, Oxford, 1975.

Aris, R., *Mathematical modelling techniques*, Pitman, London, 1978.

Denn, M. M., *Stability of reaction and transport processes*, Prentice-Hall, Englewood Cliffs, N. J., 1974.

Levine, H., *Unidirectional wave motions*, North-Holland, Elsevier, 1978.

Lin, C. C. and Segel, L. A., *Mathematics applied to deterministic problems in the natural sciences*, Macmillan, New York, 1974.

Perlmutter, D. D., *Stability of chemical reactors*, Prentice-Hall, Englewood Cliffs, N. J., 1972.

Segel, L., *Mathematics applied to continuum mechanics*, Macmillan, New York, 1977.

Shilov, G. E., *Linear spaces*, Prentice-Hall, Englewood Cliffs, N. J., 1961.