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EXISTENCE OF UNDERCOMPRESSIVE TRAVELING WAVES IN THIN FILM EQUATIONS*

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Abstract. We consider undercompressive traveling wave solutions of the partial differential equation

$$\partial_t h + \partial_x f(h) = -\partial_x(h^3\partial_x^3 h) + D\partial_x(h^3\partial_x h),$$

when the flux function f has the nonconvex form $f(h) = h^2 - h^3$. In numerical simulations, these waves appear to play a central role in the dynamics of the PDE; they also explain unusual phenomena in experiments of driven contact lines modeled by the PDE. We prove existence of an undercompressive traveling wave solution for sufficiently small nonnegative D and nonexistence when D is sufficiently large.

Key words. undercompressive shocks, traveling waves, heteroclinic orbit, existence

AMS subject classifications. 34C37, 35L65, 35L67, 35Q35, 76D08, 76D45

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1. Introduction.

The partial differential equation (PDE)

$$\partial_t h + \partial_x f(h) = -\partial_x(h^3\partial_x^3 h) + D\partial_x(h^3\partial_x h) \tag{1.1}$$

describes the flow of a thin liquid film on an inclined flat surface, under the action of gravity, viscous, and surface tension forces. Parameters governing these forces, and the slope of the surface, are incorporated into the dimensionless parameter $D \geq 0$. In particular, $D = 0$ for a vertical surface. The unknown function $h = h(x,t)$ is the (dimensionless) thickness of the thin film layer.

Equation (1.1) arises from the standard lubrication approximation of the Navier–Stokes equations [BB97, BMS99, Gre78]. We consider the specific physical problem in which the film is driven by two counteracting forces, namely, gravity pulling the film down the plane, and a thermal gradient, which induces a surface tension gradient, pushing the film up the plane. The interested reader should see [BMFC98, BMS99] for a discussion of (1.1) and the dimensionless scaling. For this particular problem, the dimensionless flux function in (1.1) is

$$f(h) = h^2 - h^3. \tag{1.2}$$

Equation (1.1) results when we assume the film height is independent of an additional transverse space variable (cf. (6.3) at the end of this paper). Experimental and numerical studies of driven contact lines [THSJ89, BB97, BMFC98, JSMB98] show that traveling wave solutions of the PDE (1.1) play an important role in the motion of the film. The significance of the nonconvexity of the flux function in (1.2) is that (1.1) then admits the possibility of undercompressive traveling waves, which we discuss in detail below.

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Driven contact line experiments that can be modeled by (1.1) show some unusual dynamics. First, there are experiments in which there is only one dominant driving force, corresponding to a convex flux function f. For example, $f(h) = h^2$ in the case of dominant Marangoni stress [CHTC90, KT97] or $f(h) = h^3$ in the case of gravitational stress [Hup82, THSJ89, dB92]. For such examples, the film forms a pronounced "capillary ridge" which corresponds mathematically to a nonmonotone traveling wave solution of (1.1). The ridge results from the interaction of surface tension, in the form of the fourth order diffusion on the right-hand side of (1.1), with the driving force, in the form of the convective term $(f(h))_x$. Such ridges have always been associated with instabilities of the film that lead to the formation of finger-like structures [BB97, THSJ89, dB92, JdB92, VIC98] in which h develops a growing oscillatory dependence on the transverse variable.

Secondly, it is interesting to contrast the driven contact line experiments where one force dominates with those experiments involving competing Marangoni and gravitational stresses. Early experiments [LL71] of relatively thick Marangoni-gravity driven films show a stable front with *monotone* decrease of the film profile from the bulk to the contact line. Recent experiments [Fan98, BMFC98] show that for intermediate thickness films, a capillary ridge forms but continues to broaden while the *contact line remains stable* and no fingering occurs. The model (1.1) with the non-convex flux (1.2) has recently been used to establish that undercompressive traveling waves are responsible for the unusual ensuing dynamics of the front [MB99, BMFC98]. In these papers, the prewetted surface is modeled as a thin precursor layer, avoiding unresolved issues of how to model a propagating liquid/solid/air contact line.

The consideration of traveling waves reduces the fourth order partial differential equation to a third order ordinary differential equation (after integrating once) depending on two parameters, namely the wave speed and the downstream film thickness. In the three-dimensional phase space of the ODE, compressive waves correspond to a codimension zero intersection of the two-dimensional unstable manifold of one equilibrium with the two-dimensional stable manifold of another equilibrium. Generically, this intersection is transverse and hence structurally stable, persisting under perturbations of the equation. For convex flux functions, e.g., $f(h) = h^2$ (Burgers flux), existence of compressive waves follows either from the analysis of Kopell and Howard [KH75] or from an argument involving a Lyapunov function and the Conley index [Mic88, Ren96, BMS99].

In contrast, undercompressive waves, which only arise when the flux is nonconvex, correspond to a codimension one intersection of the one-dimensional unstable manifold of one equilibrium with the two-dimensional stable manifold of another equilibrium. This situation typically only occurs for special values of the parameters in the ODE. The analysis of such special connections is straightforward for corresponding problems in second order ODEs; the phase space is two-dimensional and the Melnikov integral gives a measure of the separation of the manifolds in question. In our situation, the phase space is three-dimensional and the argument is more difficult. Our proof of existence of the undercompressive wave uses, in a central way, a Lyapunov function for the ODE to analyze the behavior of the one-dimensional unstable manifold from the largest equilibrium of the system. We combine this analysis with a shooting argument involving both topological properties of the orbit and quantitative estimates of higher derivatives of the solution and of its turning points. The techniques presented here apply to more general nonconvex flux functions than (1.2) and may be useful in understanding other higher dimensional bifurcation problems.

Undercompressive shock waves have been found in other physically motivated models involving systems of equations with application to dynamic phase transitions in elastic solids [AK91, Jam80, She86], liquid/vapor phase transitions [Sle83, Tru87], plane magnetohydrodynamic waves [Fre97], and multiphase flow related to secondary oil recovery [IMP90, SSMPL87, IMPT92]. Moreover, undercompressive waves have been analyzed in nonconvex conservation laws, with second order dissipation and (third order) dispersion [HL97, HS98, JMS95]. The model (1.1) represents the first realization of undercompressive shocks arising in a scalar conservation law with direct connection to experiments. The fourth order nonlinear diffusion, which has its own curious properties (see [Ber98] and references therein), combined with the nonconvex flux f yields undercompressive waves.

In an earlier paper [BMS99], we identified numerically new traveling wave solutions of (1.1) for $D = 0$ that correspond to undercompressive shock wave solutions of the conservation law. In this paper, we prove the existence of undercompressive traveling waves for small $D \geq 0$. Specifically, we show that for each downstream film thickness h_+ there is an undercompressive traveling wave, provided $D \geq 0$ is not too large. On the other hand, we also show that if D is large enough, then there is no undercompressive traveling waves with right state h_+. The latter property agrees with the limit $D \longrightarrow \infty$, for which second order diffusion dominates, and the theory is classical [Smo94]. In section 2 we discuss preliminaries concerning the phase space, and in section 3 we introduce the Lyapunov function that plays a major part in making the shooting argument work. section 4 contains the proof of existence of the undercompressive waves, while section 5 is a proof of nonexistence for large enough D.

2. Preliminaries. We are interested in traveling wave solutions of the equation

$$\partial_t h + \partial_x f(h) = -\partial_x(h^3 \partial_x^3 h) + D\partial_x(h^3 \partial_x h), \tag{2.1}$$

with $f(h) = h^2 - h^3$, and $D \geq 0$. On long scales, solutions of (2.1) behave like solutions of the corresponding scalar conservation law

$$\partial_t h + \partial_x f(h) = 0. \tag{2.2}$$

For this equation, recall that characteristics are straight lines

$$\frac{dx}{dt} = f'(h),$$

on which h is constant. A piecewise constant function

$$h(x,t) = \begin{cases} h_- & \text{if} \quad x < st, \\ h_+ & \text{if} \quad x > st \end{cases} \tag{2.3}$$

is a *shock wave solution* (with shock speed s) if the triple h_-, h_+, s satisfies the Rankine–Hugoniot condition

$$-s(h_+ - h_-) + f(h_+) - f(h_-) = 0. \tag{2.4}$$

A shock wave is *compressive* if the characteristics on each side of the shock impinge on the shock. This property is the *Lax entropy condition:*

$$f'(h_+) < s < f'(h_-). \tag{2.5}$$

As we shall see, undercompressive waves violate the Lax entropy condition.

A traveling wave solution $h = h(\xi)$, $\xi = x - st$, of (2.1) with speed s that has far field limits

$$\lim_{\xi \to -\infty} h(\xi) = h_- \text{ and } \lim_{\xi \to \infty} h(\xi) = h_+ \tag{2.6}$$

can be thought of, on large scales, as a "viscous" form of the shock (2.3). The existence of stable traveling wave profiles of (2.1) connecting the state h_- to the state h_+ is a criterion for the admissibility of the shock (2.3) in the large scale dynamics of (2.1). We are interested in the possibility of admissible undercompressive shocks, violating (2.5).

In general, traveling waves satisfy the third order ODE

$$-s(h - h_+) + f(h) - f(h_+) = -h^3 h''' + Dh^3 h'. \tag{2.7}$$

(In integrating the equation once, we have assumed $h'(\xi) \to 0$ and $h'''(\xi) \to 0$ as $\xi \to \infty$.) Equation (2.7) has two parameters $h_+ \in (0, 1/3)$ and $s > 0$. Possible left states $h = h_-$ (where $h' = 0 = h'''$) are determined by (2.4), the Rankine–Hugoniot condition for shocks.

To discuss (2.7), we begin by rewriting it:

$$h''' = g(h; h_+, s) + Dh', \tag{2.8}$$

where

$$g(h; h_+, s) = -h^{-3} \left(-s(h - h_+) + f(h) - f(h_+)\right). \tag{2.9}$$

At an equilibrium of (2.9), $h = h_e$, $g(h_e, h_+, s) = 0$, and the linearized ODE $u''' = \frac{\partial g}{\partial h}(h_e; h_+, s)u + Du'$ has characteristic equation

$$\lambda^3 - D\lambda - \frac{\partial g}{\partial h}(h_e; h_+, s) = 0. \tag{2.10}$$

For $D = 0$, the three eigenvalues are simply the three cube roots of $\frac{\partial g}{\partial h}(h_e; h_+, s)$. Since $\frac{\partial g}{\partial h}(h_e; h_+, s) = -\frac{1}{h_e^3}(f'(h_e) - s)$, the sign of $\frac{\partial g}{\partial h}(h_e; h_+, s)$ at any equilibrium h_e is related to whether characteristics at h_e, traveling with speed $f'(h_e)$, are faster or slower than the speed s of the traveling wave. For $0 \le D < 3(\frac{1}{2}\frac{\partial g}{\partial h}(h_e; h_+, s))^{2/3}$, there is one real eigenvalue $\lambda(D)$ (satisfying $\lambda(0) = (\frac{\partial g}{\partial h}(h_e; h_+, s))^{1/3}$), and two complex conjugate eigenvalues $\lambda_\pm(D)$. For larger D, all three eigenvalues are real. Moreover,

$$\lambda(D) \neq 0 \quad \text{and} \quad \mathrm{sgn}\Re(\lambda_\pm(\mathrm{D})) = -\mathrm{sgn}\lambda(\mathrm{D}) \quad \text{for all} \quad \mathrm{D}.$$

To describe the structure of equilibria, we write (2.8) as a first order system:

$$\begin{aligned} h' &= v, \\ v' &= w, \\ w' &= g(h; h_+, s) + Dv. \end{aligned} \tag{2.11}$$

We have the following classification of nondegenerate equilibria $(h, v, w) = (h_e, 0, 0)$ for (2.11).

(i) If $f'(h_e) < s$, then $\frac{\partial g}{\partial h}(h_e; h_+, s) > 0$, so that $(h_e, 0, 0)$ has a one-dimensional unstable manifold and a two-dimensional stable manifold on which, for small D, solutions spiral into the equilibrium due to the complex conjugate pair of eigenvalues with negative real part.

(ii) If $f'(h_e) > s$, then $\frac{\partial g}{\partial h}(h_e; h_+, s) < 0$, so that $(h_e, 0, 0)$ has a one-dimensional stable manifold and a two-dimensional unstable manifold on which, for small D, solutions spiral away from the equilibrium due to the complex conjugate pair of eigenvalues with positive real part.

It is convenient to label the equilibria in order of their corresponding values of h. Physically h_+ plays the role of a precursor layer in an experiment. Thus the relevant range is for h_+ small. Define $b = h_+ \in (0, 1/3)$ and let this be fixed. Treating s as a parameter, let[1] $h = h_m(s) \le h_t(s)$ be the two roots (different from h_+) of (2.4):

$$h + b - (h^2 + bh + b^2) = s$$

for s in the range

$$s_1 = f'(b) \le s \le \frac{2(f((1-b)/2) - f(b))}{(1-3b)} = s_2. \tag{2.12}$$

For brevity, we sometimes write $m = h_m(s), t = h_t(s)$. In particular (see Figure 2.1),

$$b < m < \frac{1-b}{2} < t < 1 - 2b \quad \text{if} \quad s_1 < s < s_2,$$

and

$$b = m; \quad t = 1 - 2b \quad \text{if} \quad s = s_1, \qquad m = t = (1-b)/2 \quad \text{if} \quad s = s_2.$$

Moreover, (with $h_+ = b$) the vector field (2.11) has three equilibria when $s_1 < s < s_2 : B = (b, 0, 0), M = (m, 0, 0), T = (t, 0, 0)$. From the discussion of equilibria above, we see that B and T each have a one-dimensional unstable manifold and a two-dimensional stable manifold, whereas M has a two-dimensional unstable manifold and a one-dimensional stable manifold.

The arguments of Kopell and Howard [KH75, BMS99] show that if B and M (or M and T) are sufficiently close, then there is a trajectory from M to B (or M to T, respectively). The corresponding traveling wave is necessarily compressive since $f'(b) < s < f'(m)$ (and $f'(t) < s < f'(m)$). Such trajectories lie along the intersection of the two-dimensional unstable manifold from M and the two-dimensional stable manifold from B (or T, respectively). This construction is structurally stable in that it persists under small perturbations of the vector field (for example, by changing s while keeping b fixed).

Undercompressive waves correspond to trajectories from T to B, or from B to T. These occur when the one-dimensional unstable manifold from T (from B, respectively) lies in the two-dimensional stable manifold from B (from T, respectively), a codimension one construction. The main result of section 4 is that for b fixed and for all small $D \ge 0$, there is a value of s for which there is such a trajectory from T to B. (The corresponding result from B to T follows by a symmetric argument, but is less significant physically.)

In section 5 we show that for each $b < 1/3$, and for D sufficiently large, there is no value of s for which there is an undercompressive traveling wave from T to B. This result expresses the notion that for large D, second order diffusion dominates fourth order diffusion. In the absence of fourth order diffusion, the only traveling waves are compressive.

[1]Note that the subscript t here does not denote partial derivative. It is an index to denote the specific equilibrium.

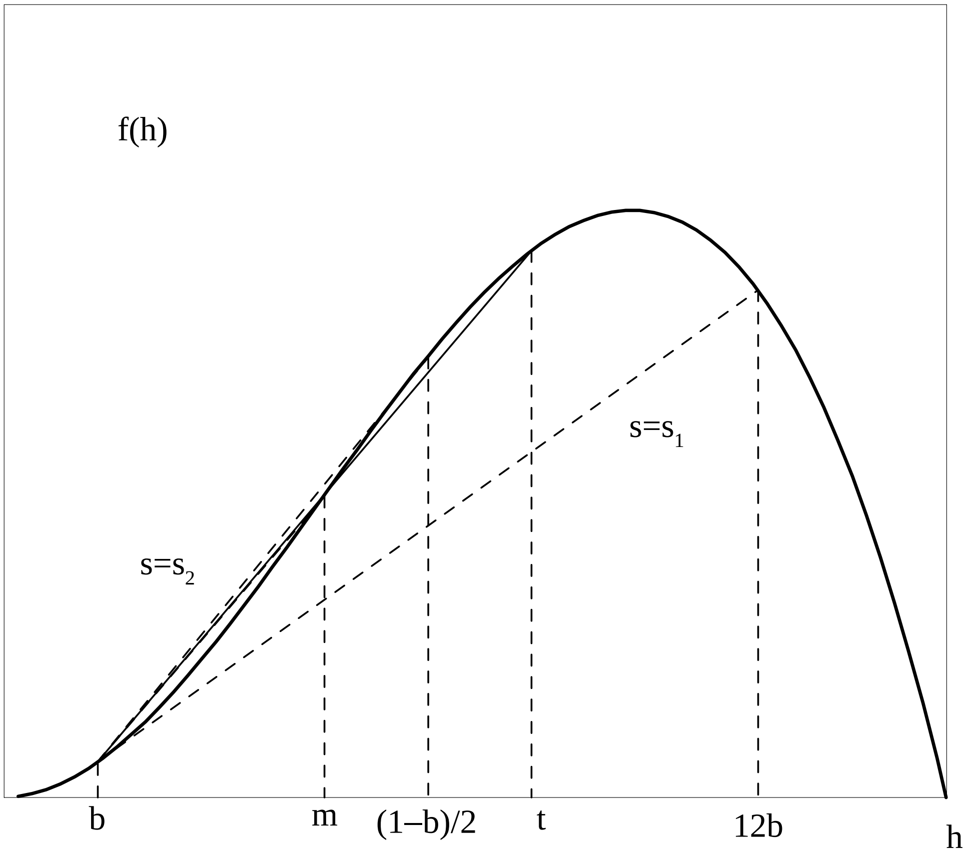

FIG. 2.1. *Flux function $f(h) = h^2 - h^3$, and chords indicating equilibria and wave speeds.*

3. The Lyapunov function. Equation (2.8) has a Lyapunov function

$$L(h) = h''h' + R(h),$$

where

$$\frac{dR}{dh}(h) = -g(h; b, s),$$

which we use extensively in the analysis of traveling waves. The equilibria B, M, T correspond to extrema b, m, t of $R(h)$, as shown in Figure 3.1.

Differentiating along a solution $h(\xi)$ and using the ODE (2.8), we find that

$$L(h)' = (h'')^2 + D(h')^2.$$

Therefore, $L(h)$ increases along trajectories. In particular, $R(h)$ increases at successive critical points of a solution $h(\xi)$ of (2.8). It follows that for any traveling wave solution connecting extrema of $R(h)$ there exist a priori upper and lower bounds for the critical

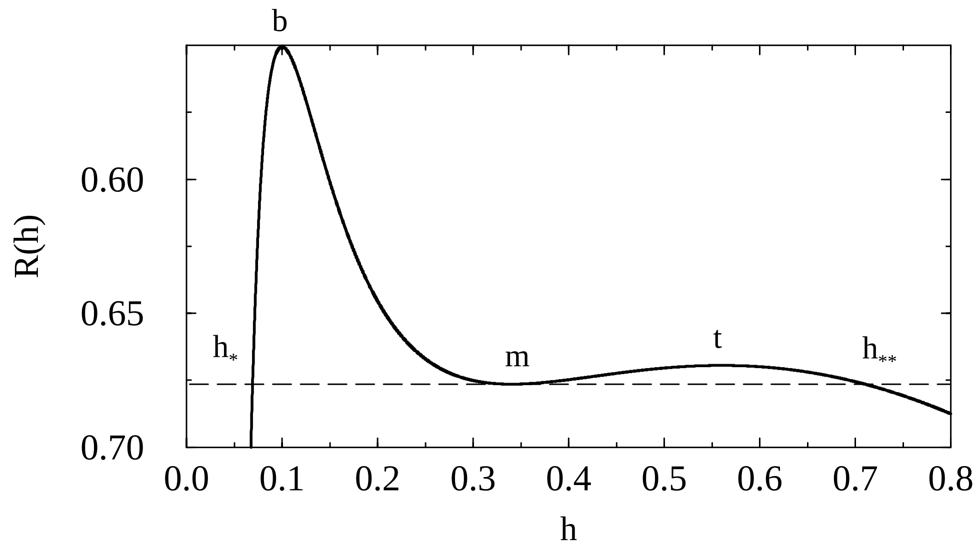

FIG. 3.1. *The function $R = -dg/dh$ in the Lyapunov function. Pictured are the three equilibria, b, m, and t, of* (2.11) *and the a priori upper and lower bounds h_{**} and h_*, defined in* (3.2)*, for a traveling wave solution.*

points h_{crit} of the traveling wave

$$h_* < h_{crit} < h_{**}, \tag{3.1}$$

where h_* and h_{**} are defined by (see Figure 3.1)

$$h_* = \min\{h : R(h) \geq R(m)\}, \quad h_{**} = \max\{h : R(h) \geq R(m)\}. \tag{3.2}$$

Note that R and hence h_*, h_{**} depend on b and s.

4. Existence of an undercompressive wave. In this section, we fix $h_+ = b < 1/3$, and consider the vector field (2.11) with s and D varying. An undercompressive wave occurs when there exists a trajectory (a *heteroclinic orbit*) from the equilibrium $T = (h_t(s), 0, 0)$ to the equilibrium $B = (b, 0, 0)$. We show that for sufficiently small D (depending on the value of b), there exists a special value of s, call it s_*, for which there is such a trajectory.

For each value of s, we consider a special solution of (2.11), corresponding to the branch of the unstable manifold from the equilibrium $(h_t(s), 0, 0)$ that initially decreases in h. Let $h_t(\xi; s)$, $-\infty < \xi$, denote the solution of (2.8) corresponding to this branch. By the stable manifold theorem and the Picard continuation theorem for ODEs, $h_t(\xi; s)$ is smooth and tangent to the unstable manifold of the linearized ODE about h_t and is determined uniquely up to translation in ξ. The goal of this section is to prove that there is an s_* for which

$$\lim_{\xi \longrightarrow \infty} h_t(\xi; s_*) = b.$$

The proof of this result was inspired in part by results from numerical simulations [BMS99, M99]. The argument is a one parameter, one direction shooting argument. In Proposition 4.1 below, we show that for s near s_1, for which b and $h_m(s)$ are close, $h_t(\xi; s)$ decreases monotonically, reaching zero at a finite value of ξ. On the other hand, in Proposition 4.2, we show that for s near s_2, so that $h_m(s)$ and $h_t(s)$ are close, $h_t(\xi; s)$ has a minimum value above $h = b$; the trajectory then increases without bound. The trajectory we seek lies between these two extremes; its existence is established in Theorem 4.8. While this paper establishes existence of such a special shock speed, uniqueness remains an open problem. However, numerical computations [BMS99, M99] reveal the shock speed and undercompressive wave to be unique.

The first part of this argument is based on the Lyapunov function. First note that there is a value of s, call it s_l, such that the two maxima of $R(h; s)$, b and $h_t(s)$ satisfy $R(b; s) = R(h_t(s); s)$ and

$$R(b; s) < R(h_t(s)) \quad \text{for all} \quad s \quad \text{satisfying } f'(b) < s < s_l.$$

The function R has a global maximum at $h_t(s)$ for s in this range. Figure 3.1 shows a case where $s > s_l$; in this case R has a global maximum at b.

PROPOSITION 4.1. *For all $s, f'(b) < s < s_l$, $h_t(\xi; s)$ decreases monotonically to hit zero at a finite value of ξ.*

Proof. Suppose that $h_t(\xi; s)$ has a local minimum at a finite $\xi = \xi_0$. Then, necessarily $v_t(\xi; s) \equiv (h_t(\xi; s))_\xi$ is zero at ξ_0. Since the Lyapunov function increases along trajectories, we have $R(h_t(\xi_0; s) \geq R(h_t(s))$. However, this contradicts the fact that $h_t(s)$ is a global maximum of R for this range of s. Thus there can be no local minimum at finite ξ_0.

Now we show the solution decreases to hit zero at finite ξ. To see this, we note that since h is monotonically decreasing, it either hits zero at finite ξ_0 or it stays positive for all $\xi < \infty$, which means that because it is decreasing, it has a limit $h \to h_0 \geq 0$. We now show that the latter case leads to a contradiction.

First suppose that $h_0 > 0$. Then the only choices are $h_0 = b$ or $h_0 = h_m(s)$. Otherwise, (2.8) implies that $h''' - Dh'$ will remain bounded away from zero on an interval of the kind $[l_0, \infty)$ which implies that $h'' - Dh$, and hence h'' and h' become unbounded, which is a contradiction. The two equilibria, b and $h_m(s)$ are also ruled out by the properties that $L(h)$ is an increasing function of ξ, $L = R$ at equilibria, and R has a global maximum at $h = h_t(s)$. Thus we can not have that $h_0 > 0$.

Now suppose h decreases monotonically to zero in infinite time. Again, from the ODE, this implies that eventually the $h''' - Dh'$ becomes monotonically unbounded, inconsistent with h decreasing monotonically to zero.

The only choice then is for $h \to 0$ at some finite ξ. ∎

PROPOSITION 4.2. *Let $b \in (0, 1/3)$. There are numbers $D_0, \underline{s}, \overline{s}$ with $s_\ell < \underline{s} < \overline{s} < s_2$ such that for all $D \in [0, D_0]$ and all $s \in [\underline{s}, \overline{s}]$, $h_t(\xi, s)$ has a global minimum between $h_m(s)$ and b. The solution then increases without bound after reaching that minimum.*

Proof. It suffices to prove that $h_t(\xi; s)$ has a global minimum between $h_m(s)$ and b. The result then follows from the Lyapunov function and a similar argument to the first part of the proof of Proposition 4.1.

First we prove that for $D = 0$, there is a range of s, $s_u < s < s_2$ for which $h_t(\xi, s)$ has the property claimed in the proposition. Then we use a perturbation argument to prove the result for small positive D.

To show that $h_t(\xi; s)$ has such a minimum, we first estimate the trajectory at $h_m(s)$ in terms of the parameter $\rho = h_t(s) - h_m(s)$, which decreases to zero as s approaches s_2. In what follows, we consider $\rho > 0$ to be small.

LEMMA 4.3. *Let $D = 0$. Then $h_t(\xi_m; s) = h_m(s)$ for some $\xi_m < \infty$.*

Proof. To simplify notation, consider s fixed, and write $h = h(\xi)$ in place of $h = h_t(\xi, s)$. Then h has the properties

$$(h, h', h'') \longrightarrow (h_t(s), 0, 0) \quad \text{as} \quad \xi \longrightarrow -\infty,$$

and moreover

$$(h, h', h'') \sim (h_t(s), 0, 0) - Ce^{\lambda\xi}(1, \lambda, \lambda^2) \quad \text{as} \quad \xi \longrightarrow -\infty,$$

where

$$\lambda = \left(\frac{\partial g}{\partial h}(h_t(s); b, s)\right)^{1/3}$$

is the positive eigenvalue for the equilibrium $(h_t(s), 0, 0)$ for (2.11). In particular, for ξ sufficiently negative,

$$h(\xi) < h_t(s), v(\xi) = h'(\xi) < 0, w(\xi) = h''(\xi) < 0. \tag{4.1}$$

Now define the open set $O \subset R^3$ by

$$O = \{(h, v, w) : h_m(s) < h < h_t(s), v < 0\}.$$

Then $(h, v, w)(\xi) \in O$ for all $\xi < M$, for some M.

Next note that the vector field (2.11) is uniformly Lipschitz in O, since the only nonlinearity is in $g(h; b, s)$, a function of h alone (for fixed b, s) whose derivative is bounded for $h_m(s) \leq h \leq h_t(s)$. Therefore, the solution $(h, v, w)(\xi)$ can be continued in ξ as long as it remains in O. That is, either (a) the solution stays in O for all $\xi \in R$, or (b) the solution exits S at some finite $\xi = \xi_T$.

In case (a), the solution must approach an equilibrium as $\xi \longrightarrow \infty$. Thus, $(h, v, w) \longrightarrow (h_m(s), 0, 0)$, or $(h, v, w) \longrightarrow (h_t(s), 0, 0)$. Both possibilities are ruled out by the property that the Lyapunov function must increase along the trajectory, since $R(h_m(s))$ and $R(h_t(s))$ are both less than or equal to the value of the Lyapunov function at $\xi = -\infty$.

In case (b), $(h, v, w)(\xi_T) \in \partial O$. Thus, (i) $h'(\xi_T) = 0$, or (ii) $h(\xi_T) = h_t(s)$, or (iii) $h(\xi_T) = h_m(s)$. But in O, $h''' = g(h; b, s) < 0$, so that h'' is decreasing, hence negative, by (4.1). But this implies that h' is decreasing, and must also be strictly negative, contradicting (i) and ruling out (ii). Hence, (iii) holds, completing the proof of the lemma. $\square$

To proceed further with the proof of Proposition 4.2, we parameterize the unstable manifold

$$\{(h, v, w)(\xi) : -\infty < \xi < \infty\}$$

by h, up to the first minimum of $h(\xi)$. That is, we consider $v = v(h) = h'(\xi)$, and we write (2.8) with $D = 0$ as a nonautonomous equation for $v(h)$:

$$v^2 v'' + v(v')^2 = g(h; b, s). \tag{4.2}$$

The solution of (4.2) we consider satisfies $v(t) = 0$, and $v'(t) > 0$. In fact, higher derivatives of v at $h = t$ can easily be determined using the Taylor series of $v(h)$ about

$h = t$. To simplify notation, we write $h_t(s) = t$, and $h_m(s) = m$. Then $\rho = t - m > 0$. Next, define a new function $G(h;\rho)$ for $\rho \geq 0, t-\rho \leq h \leq t$ by

$$G(h,\rho) = \begin{cases} g(h;b,s)/[(h-t)(h-t+\rho)] & \text{for} \quad t-\rho < h < t, \\ -\dfrac{1}{\rho}\dfrac{\partial g}{\partial h}(t-\rho;b,s) & \text{if} \quad h = t-\rho, \\ \dfrac{1}{\rho}\dfrac{\partial g}{\partial h}(t;b,s) & \text{if} \quad h = t. \end{cases}$$

Then G is as smooth as g, except at $h = t, h = t - \rho$, where, in general, G loses a derivative. For the specific g in this paper, both g and G are rational functions of h, so there is no loss of derivative. Note that $G(h,\rho) > 0 \quad \text{for} \quad t-\rho \leq h \leq t$, and $g(h;b,s) = (h-t)(h-t+\rho)G(h,\rho)$.

We scale (4.2) as follows: Write

$$h = t + \rho\theta, \quad v = \rho^{4/3}y(\theta). \tag{4.3}$$

Then (4.2) becomes

$$y^2y'' + yy'^2 = \theta(\theta+1)G(t+\rho\theta,\rho). \tag{4.4}$$

From the Taylor series expansion of $y(\theta)$ about $\theta = 0$, where we impose the condition $y(0) = 0$, we find

$$y'(0) = G(t,\rho) = G(t,0) + O(\rho). \tag{4.5}$$

Now Lemma 4.3 implies the following.

LEMMA 4.4. *There are constants $\rho_0 > 0, 0 < \alpha < \beta$, such that for each $\rho \in (0,\rho_0)$, the solution $y(\theta), -1 \leq \theta \leq 0$, of* (4.4) *satisfying $y(0) = 0$,* (4.5) *also satisfies*

$$-\beta < y(-1) < -\alpha, \quad \alpha < y'(-1) < \beta.$$

Proof. First note that $\theta = -1$ corresponds to ξ_m in Lemma 4.3. The scaled Lyapunov function is $L(y) = y^2y' + R(\theta)$, where

$$R(\theta) = \int_\theta^0 \eta(\eta+1)G(t+\rho\eta,\rho)d\eta < 0$$

for $-1 \leq \theta < 0$. Specifically, $\frac{d}{d\theta}L(y) = yy'^2 < 0$, so that $L(y)$ increases as θ, hence y decreases. If $y' = 0$, then $L(y) = R(\theta) < R(0) = 0 = L(0)$. Thus, $L(y)$ has not increased, which is a contradiction. Therefore, $y' > 0$ along the entire trajectory from $\theta = 0$ to $\theta = -1$. The result now follows by bounding the $O(\rho)$ term in G. ∎

From the rescaling (4.3) and the chain rule, we conclude the following.

COROLLARY 4.5. *Let $v(h)$ be the solution of* (4.2) *corresponding to $y(\theta)$ of Lemma* 4.4. *Then*

$$-\rho^{4/3}\beta < v(m) = \rho^{4/3}y(-1) < -\rho^{4/3}\alpha; \quad \rho^{1/3}\alpha < \frac{dv}{dh}(m) = \rho^{1/3}\frac{dy}{d\theta}(-1) < \rho^{1/3}\beta. \tag{4.6}$$

Now, for $b \le h \le m$, $g(h) > 0$, so that as long as $v(h) < 0$, (4.2) implies that $v''(h) > 0$. Thus v is a convex function of h whenever v is negative. We use this fact below.

To complete the proof of Proposition 4.2, we suppose that $v(h) < 0$ for $b < h < m$ and look for a contradiction. The idea is to show that for small enough ρ, by estimating v and $v'(h)$ over half this interval, when we integrate (4.2) the integral of g remains bounded away from zero, while the integral of the left-hand side approaches zero.

Note that for all $b < h < m$,

$$0 > v(h) = v(m) - \int_h^m v'(\eta)d\eta \ge v(m) - (m-h)v'(m)$$

(since $v'(\eta) \le v'(m)$ for $\eta \le m$). Therefore,

$$|v(h)| \le |v(m)| + (m-h)K_1\rho^{1/3} \le K\rho^{1/3}, \quad h_M \le h \le m,$$

for some $K > 0$ independent of ρ.

Also, the convexity of v and inequality (4.6) imply that

$$K\rho^{1/3} > v'(m) > v'(h) \ge \frac{v(h) - v(b)}{h-b} \tag{4.7}$$

for all $b < h < m$. Now consider $h_M = (m+b)/2$. The above inequalities imply that

$$|v(h_M)| \le K\rho^{1/3}, \qquad |v'(h_M)| \le K_1\rho^{1/3},$$

where in the second inequality we use (4.7), the bounds on v, and the fact that h_m is not close to b.

Now we integrate (4.2) from h_M to m, integrating the left-hand side by parts:

$$v(m)^2v'(m) - v(h_M)^2v'(h_M) - \int_{h_M}^m v(h)v'(h)^2dh = \int_{h_M}^m g(h)dh.$$

But the left-hand side is order ρ, while the right-hand side is order one, as $\rho \longrightarrow 0$. This contradiction implies that $v(h) = 0$ for some $h \in (b, m)$, for each $\rho > 0$ sufficiently small.

To summarize, we have so far shown that for $D = 0$, there is a range $s_u < s < s_2$ for which the unstable manifold from t decreases to a global minimum between m and b and then increases without bound. To continue the proof of Proposition 4.2, we need to establish the same behavior for small $D > 0$. Since the unstable manifold from t depends continuously on D, away from $s = s_2$ (at $s = s_2$, two equilibria coincide, so the unstable manifold degenerates), there is $D_0 > 0$ and two values of s, say $s_u < \underline{s} < \overline{s} < s_2$ such that for $0 \le D \le D_0$, $\underline{s} \le s \le \overline{s}$, the unstable manifold from t has $h_\xi = v$ changing sign for h between m and b. It then follows from the Lyapunov function argument used previously that the solution $h(\xi; s)$ has a global minimum between m and b.

Finally we note that the solution increases without bound after the local minimum between m and b. This is because, like the preceding arguments based on the Lyapunov function, the solution cannot have a local maximum after hitting this minimum and cannot asymptote to either the fixed point m or b. This completes the proof of Proposition 4.2. $\square$

We now define two distinguished values of h. Let

$$\overline{h} = \max_{s_1 \leq s \leq s_2} h_{**}(s), \quad \underline{h} = \min_{s_1 \leq s \leq s_2} h_*(s), \tag{4.8}$$

where $h_* = h_*(s), h_{**} = h_{**}(s)$ are given by (3.2).

LEMMA 4.6. *For all $s \in (s_1, s_2)$, the trajectory $h_t(\xi, s)$ crosses the boundary of the set $\overline{h} < h < \underline{h}$ at most once, either by increasing h above $\underline{h}$ or by decreasing h below $\overline{h}$. In the former case, the solution increases without bound after it leaves this set and in the latter case, the solution hits zero at finite ξ.*

Proof. Suppose the trajectory $h_t(\xi, s)$ crosses the lower boundary $\underline{h}$. Then it is impossible for the solution to turn around. If it did, there would be a local minimum at a value $h_{\min} < \underline{h}$, which by the definition of $\underline{h}$ in (4.8) violates the Lyapunov condition (3.1). Likewise if $h_t(\xi, s)$ crosses the upper boundary $\overline{h}$ it cannot turn around because this would again violate (3.1). □

Now define

$$S = \{s \in (s_1, s_2) | h_t(\xi, s) \text{ increases above } \overline{h} \text{ for some finite } \xi\}.$$

For all D satisfying the conditions of Proposition 4.2 we know that S is not empty; it contains at least one interval near s_2. Also, from Proposition 4.1, we know that S does not contain any $s < s_l$. Thus for all D satisfying the conditions of Proposition 4.2, the following special value of s is well defined:

$$s_* = \inf\{S\}. \tag{4.9}$$

Clearly $s_* \geq s_l$.

LEMMA 4.7. *Let D satisfy the conditions of Proposition* 4.2 *and s_* be defined as in* (4.9)*. Then the trajectory $h_t(\xi; s_*)$ remains bounded between $\overline{h}$ and $\underline{h}$ and can be continued in this range for all $\xi < \infty$.*

Proof. First we note that $h_t(\xi; s_*)$ stays below $\overline{h}$. Suppose it crosses $\overline{h}$ at finite $\xi = \xi_0$ (i.e., $s_* \in S$). Since solutions of (2.11) have continuous dependence on the parameter s, there then exists an $\epsilon > 0$ so that $s_* - \epsilon' \in S$ for all $0 < \epsilon' < \epsilon$. This contradicts the fact that $s_* = \inf S$. Now we show that $h_t(\xi; s_*)$ stays above $\underline{h}$ for all ξ. Suppose it does not. Then there exists a value ξ_0 at which $h_t(\xi; s_*)$ crosses $\underline{h}$. Again, since solutions of (2.11) have continuous dependence on the parameter s, there exists an $\epsilon > 0$ so that for all $\epsilon > \epsilon' > 0$, $h_t(\xi; s_* + \epsilon')$ crosses the lower bound $\underline{h}$. Hence $s_* + \epsilon' \notin S$. However, this contradicts the fact that s_* is the infimum.

Thus the trajectory $h_t(\xi; s_*)$ is guaranteed to stay between $\underline{h}$ and $\overline{h}$. We need to show that the trajectory can be continued for all time. As in the proof of Lemma 4.3, we can do this by using the continuation part of the Picard theorem for ODEs, provided we can show uniform Lipschitz continuity of $(v, w, g(h))$ as a function of (h, v, w). Since the solution is guaranteed to have h bounded between $\underline{h}$ and $\overline{h}$, by the form of g, Lipschitz continuity is guaranteed for the third component. Moreover, the other two terms are linear in v and w, so that uniform Lipschitz continuity is guaranteed for (h, u, v) in the set $[\underline{h}, \overline{h}] \times R \times R$. The solution can thus be continued for all $\xi < \infty$. □

THEOREM 4.8. *Given D satisfying the conditions of the statement of Proposition* 4.2*, and s_* defined in* (4.9)*, the unstable manifold $h_t(\xi, s_*)$ (with s_* defined as above) connects the equilibrium $h_t(s_*)$ to the equilibrium b and hence describes an undercompressive wave.*

Proof. We have that $h_t(\xi, s_*)$ is bounded. For ease of notation below, we denote this trajectory simply by $h(\xi)$.

Case 1: There exists a finite $\xi_{\max}$ above which $h(\xi)$ has no extrema, i.e., it is monotone increasing or decreasing. Since h is bounded, it is convergent to a limit as $\xi \to \infty$. That limit must be an equilibrium. If not, then h''' is uniformly bounded away from zero on a semi-infinite line $[\xi_0, \infty)$ and we can show this causes h, h', and h'' to become unbounded. That equilibrium has to be either m, b, or t. However, by comparing the values of the Lyapunov function, we see that the only choice is b, since $R(m) < R(h_t)$, the Lyapunov function can be shown to initially increase by comparing the solution with the predicted linear theory. Note that although this case does imply that $h(\xi) \to b$ as $\xi \to \infty$, this case is not the expected scenario. Note that when D is small, the stable manifold of b has two complex conjugate eigenvalues so that we would expect a trajectory on it to spiral in to b, i.e., we expect such a solution to have an infinite number of local extrema as $\xi \to \infty$.

Case 2: There exists a set of points $X \in R$ where h has an extremum and hence $h_\xi = 0$, and $\sup\{X\} = \infty$.

First note that such points are isolated. This is because the a priori upper and lower bounds on the solution and the fact that it satisfies (2.8) imply that $h(\xi)$ is a global real analytic function, and hence if there is a cluster point for $h_\xi = 0$, then h_ξ must be identically zero, which is clearly not the case. Thus the set X must be a countable set ξ_i and $\xi_i \to \infty$ as $i \to \infty$.

Denote by h_i the value $h(\xi_i)$. Let us suppose without loss of generality that ξ_i are local minima for i odd and maxima for i even. From the Lyapunov function, we see that all extrema satisfy $R(h(\xi_i)) > R(h_t)$. Moreover $R(h_i)$ is an increasing sequence that is also bounded, so it converges to a value R_1. Furthermore, all minima must lie below b. This is because if, say, h_k lies above b, then since $R(h)$ is monotone decreasing on the set $R(h) > R(h_t)$, $h > b$, then $R(h_{k+1}) < R(h_k)$ because h_{k+1} is a local maximum. However, this contradicts the fact that $R(h_i)$ is increasing. Likewise, a similar argument shows that the local maxima all lie above b.

By (3.1) and (4.8), the solution lies between $\underline{h}$ and $\overline{h}$. The proof follows provided we can show that h' and h'' approach zero as $\xi \to \infty$.

To do this, we make some explicit estimates, using (2.8) and the Lyapunov function. First note that since $R(h_i)$ is increasing, the h_i oscillate around $h = b$: $h_i < b$ at a min and $h_{i+1} > b$ at a max. Therefore, there are two convergent subsequences, $h_{2n+1} \to h_1$, $h_{2n} \to h_2$, with $h_2 - h_1 \geq 0$.

Now note that for each ξ_i,

$$R(h_i) - R(h_t) = \int_{-\infty}^{\xi_i} h_{\xi\xi}^2 + D \int_{-\infty}^{\xi_i} h_\xi^2.$$

Taking the limit as $i \to \infty$ and recalling that $D \geq 0$ gives

$$\int_{-\infty}^{\infty} h_{\xi\xi}^2 d\xi \leq D \int_{-\infty}^{\infty} h_\xi^2 d\xi + \int_{-\infty}^{\infty} h_{\xi\xi}^2 d\xi = R_1 - R(h_t) \leq R(b) - R(h_t) < \infty.$$

We now invoke the following interpolation inequality [Tay96, p. 9];

$$\|h_\xi\|_{L^4(R)} \leq C \|h\|_{L^\infty}^{1/2} \|h_{\xi\xi}\|_{L^2(R)}^{1/2}.$$

This means that since h is uniformly bounded and $h_{\xi\xi}$ is bounded in $L^2(R)$ that h_ξ is bounded in $L^4(R)$.

On $[\xi_i, \xi_{i+1}]$, h_ξ has the fixed sign $(-1)^{i+1}$. Now choose $\beta_i \in [\xi_i, \xi_{i+1}]$ so that $|h_\xi|$ attains a maximum on this interval at β_i. Compute

$$\begin{aligned} |h_\xi^3(\beta_{i+1}) - h_\xi^3(\beta_i)| &= |h_\xi(\beta_{i+1})|^3 + |h_\xi(\beta_i)|^3 \\ &= 3\left|\int_{\beta_i}^{\beta_{i+1}} h_\xi^2 h_{\xi\xi} d\xi\right| \\ &\le 3\int_{\beta_i}^{\beta_{i+1}} |h_\xi|^2 |h_{\xi\xi}| d\xi \\ &\le 3\left[\int_{\beta_i}^{\beta_{i+1}} |h_\xi|^4\right]^{1/2} \left[\int_{\beta_i}^{\beta_{i+1}} |h_{\xi\xi}|^2 d\xi\right]^{1/2} \\ &\le 3\epsilon_i\delta_i, \end{aligned}$$

where

$$\delta_i = \left[\int_{\beta_i}^{\beta_{i+1}} |h_\xi|^4\right]^{1/2}, \quad \epsilon_i = \left[\int_{\beta_i}^{\beta_{i+1}} |h_{\xi\xi}|^2 d\xi\right]^{1/2},$$

and where $\sum_i \epsilon_i^2$ and $\sum_i \delta_i^2$ are both finite. Thus $\epsilon_i\delta_i \to 0$ as $i \to \infty$. By the choice of β, this also implies that $|h_\xi|^3$ and hence $|h_\xi|$ goes to zero as $\xi \to \infty$. Note that this also implies that h_ξ is uniformly bounded on R.

We now show that $h_{\xi\xi}$ is uniformly bounded independent of ξ. Since h solves the ODE (2.11), and since h is uniformly bounded between $\underline{h}$ and $\overline{h}$, we have that $h_{\xi\xi\xi} - Dh_\xi$, and hence $h_{\xi\xi\xi}$ is uniformly bounded. Thus for any ξ,

$$|h_{\xi\xi}^3(\xi)| = 3|\int_{-\infty}^{\xi} h_{\xi\xi}^2 h_{\xi\xi\xi} d\xi| \le C\|h_{\xi\xi}\|_{L^2}^2 < \infty.$$

Finally note that the Lyapunov function is $h_\xi h_{\xi\xi} + R(h)$. Since it is increasing, and the product $h_\xi h_{\xi\xi}$ goes to zero as $\xi \to \infty$, then $R(h(\xi))$ approaches a constant. The infinite sequence of alternating max and mins implies that that constant has to be $R(b)$.

Finally, since the trajectory $\{(h, h', h'')(\xi) : -\infty < \xi < \infty\}$ is bounded, and there are no periodic orbits, the trajectory must approach an equilibrium as $\xi \longrightarrow \infty$. The equilibrium is necessarily b, since this is the only equilibrium with $R(h) > R(t)$. This completes the proof of Theorem 4.8 $\square$

5. Nonexistence of undercompressive waves for large *D*. In this section, we show that for each $b < 1/3$, and for D sufficiently large, there are no undercompressive traveling wave solutions having $h = b$ as the downstream height. This is formulated precisely in the following theorem, in which, as in the previous section, we fix $h_+ = b$, and consider the vector fields (2.11) to be parameterized by s and D.

THEOREM 5.1. *Let $b \in (0, 1/3)$. Then there is $D_1 > 0$ such that for $D > D_1$ and $s_1 < s < s_2$, there is no orbit from the equilibrium $(h_t(s), 0, 0)$ to the equilibrium $(b, 0, 0)$.*

Proof. We need to show that the unstable manifold from $h_t(s)$ never connects to the fixed point at b.

Recall the ODE is

$$
\begin{aligned}
h' &= v, \\
v' &= w, \\
w' &= g(h) + Dv,
\end{aligned} \tag{5.1}
$$

where

$$
g(h) = -\frac{f(h) - sh - f(b) + sb}{h^3}.
$$

Some of the results in the preceding section apply to this case, in particular Proposition 4.1 and Lemma 4.6. Using these two results, we now show that for sufficiently large D (depending on b) for all s in the region that we are interested in, h_t decreases monotonically to zero, in which case it can never connect to the fixed point at b.

To see that this is true, first note that the linearization of (5.1) near the fixed point $h_t(s)$ yields eigenvalues that satisfy the equation

$$
\lambda^3 - \lambda D - g'(h_t) = 0.
$$

Since $g'(h_t) \geq 0$, for small D there is one positive real root and two complex roots with negative real part. For sufficiently large D there is one positive real root $\lambda_p \sim \sqrt{D}$ and two real negative roots $\lambda_1 \sim -g'(h_t)/D$ and $\lambda_2 \sim -\sqrt{D}$.

First we note that the Lyapunov function guarantees that the branch of the unstable manifold from h_t that initially increases can not turn around to connect to b. This is because if the solution turns around, it must have a local maximum above h_t; however, the function $R(h)$ decreases monotonically above h_t.

Consider now the branch of the unstable manifold from h_t that initially decreases. We show that for D sufficiently large, this branch decreases to zero at finite ξ.

To linear order the solution looks like

$$
h_t(\xi; s) = h_t - e^{\lambda_p \xi} \tag{5.2}
$$

for ξ very negative. Also, to linear order,

$$
v \sim \lambda_p (h - h_t),
$$

and as long as $h > \underline{h}$ (defined in (4.8)) we have an a priori bound for $g(h)$. In particular, we can choose D large enough so that the ODE (5.1) is dominated by the linear behavior (i.e., $g = 0$) of the ODE while $h > \underline{h}$. However, the linear behavior simply has that h decreases monotonically like (5.2). So for D large enough, the solution should decreases monotonically until it hits $\underline{h}$. However, we know that once it hits $\underline{h}$ it continues to decrease by Proposition 4.1.

We now make the above argument rigorous. Introducing the new variables

$$
Q = \frac{v}{h - h_t}, \quad P = \frac{w}{h - h_t},
$$

the ODE (5.1) is transformed to the system

$$
\begin{aligned}
Q' &= P - Q^2, \\
P' &= B(h(\xi)) + DQ - QP,
\end{aligned} \tag{5.3}
$$

$$
\text{where } B(h) = \frac{g(h)}{h - h_t}.
$$

Note that since g vanishes at h_t, B is bounded and approaches g' at h_t. Also, for $\underline{h} < h < h_t$, $|B|$ is bounded independent of D. Call this bound $M(b)$. This system has a fixed point for $h = h_t$ that corresponds to the positive eigenvalue λ_p above, $Q = \lambda_p$, $P = \lambda_p^2$. Now consider the rectangle

$$R_D = \{(Q, P) | \sqrt{D}/2 < Q < 2\sqrt{D} \text{ and } D/2 < P < 2D\}. \tag{5.4}$$

Choose D to satisfy $D > (4M)^{2/3}$. Then as long as $\underline{h} < h < h_t$, on the boundary of R_D, the vector field in (5.3) points into R_D, which means that the solution remains in R_D. This gives a lower bound on $Q = \frac{h_\xi}{h - h_t}$,

$$\frac{h_\xi}{h - h_t} \geq K > 0,$$

which implies $h_\xi < K(h - h_t)$ so that $h(\xi)$ decreases exponentially: for all $\xi > \xi_0$, $h(\xi) - h_t \leq (h(\xi_0) - h_t)e^{K(\xi - \xi_0)}$ provided $\underline{h} < h(\xi') < h_t$ for all $\xi' < \xi_0$. By the stable manifold theorem, we know there exists such an ξ_0 where $\underline{h} < h(\xi_0) < h_t$. This is sufficient to guarantee that $h(\xi)$ hits $\underline{h}$ at a finite value of ξ. ∎

6. Summary and conclusions. We have considered traveling wave solutions $h(x - st)$ of the PDE

$$\partial_t h + \partial_x(h^2 - h^3) = -\partial_x(h^3 \partial_x^3 h) + D\partial_x(h^3 \partial_x h). \tag{6.1}$$

Recent numerical experiments [BMS99, M99] show that certain jump initial data give rise to undercompressive structures, in which the leading part of the structure is an undercompressive traveling wave, connecting states h_- to h_+, for which the speed s of the wave violates the Lax entropy condition

$$f'(h_+) < s < f'(h_-).$$

For a fixed value of h_+, the numerics show a special value of h_- for which an undercompressive waves exists when the parameter D in (6.1) is small. Likewise, for large D, the numerics show that undercompressive waves do not exist. In this paper we presented rigorous proofs of both of these numerical observations.

Traveling waves satisfy a third order autonomous ODE in which the downstream thickness h_+ and the wave speed s appear as parameters. For each $h_+ = b < 1/3$, there is a range of s for which the ODE has three (hyperbolic) equilibria, B, M, and T. M has a two-dimensional unstable manifold while B and T have two-dimensional stable manifolds. Compressive waves are heteroclinic orbits from M to either B or T, codimension zero intersections of a two-dimensional stable manifold from one fixed point with a two-dimensional unstable manifold from another fixed point. Such intersections are structurally stable to perturbations and exist for a range of the parameter s. In contrast, undercompressive waves are heteroclinic connections from either T (or B) to B (or T, respectively). The situation that corresponds to the physical problem of interest is the existence of a wave from T to B.

Our analysis relies heavily on the existence of a Lyapunov function for the ODE. It follows directly from the Lyapunov function that there is a range of s (where M is close to B) for which a branch of the unstable manifold from T decreases monotonically to zero. We then consider a range of s for which M is very close to T and rescale the ODE using the distance from M to T as a scaling parameter. By analyzing the rescaled equation for $D = 0$, we are able to show that whenever M is sufficiently close to T, the initially decreasing branch of the unstable manifold from T has a global minimum (in h) between m and b. Moreover, a perturbation argument shows this property for $D \geq 0$ and small, provided M is not too close to T. We then proceed with an argument that shows that there is an intermediate value of the parameter s, so that M is neither very close to B or to T for which the unstable manifold from T must connect to B. This part of the proof is largely topological, but it includes some explicit estimates on higher derivates of the solution along the unstable manifold from T in order to guarantee that it stays bounded and hence connects to B.

In the last section of the paper, we show that for large values of D, regardless of the speed s of the wave, the unstable manifold from T never connects to B. The result is that there can never exist an undercompressive wave. The proof follows from making a change of variables in the ODE to show that the linear system dominates the dynamics along the unstable manifold, until the solution gets so small in h that it must hit zero at finite ξ.

We mention some related papers discussing third order (ODE) travelling waves that exist only for special parameter values or wave speeds. The paper of Grinfeld [Gri89] deals with travelling waves for Korteweg capillary regularization of a van der Waals fluid and uses Conley index theory to prove existence. The paper [BHP96] deals with traveling waves in the compressive case $f(h) = h^3$ but with a different form of degenerate diffusion. They prove existence of waves with a sharp contact line (h goes to zero) using a two directional shooting method. It would be interesting to see if the methods of these papers also apply to the problem presented here.

It is interesting to note that our arguments extend, with slight modifications, to the case of linear diffusion:

$$\partial_t h + \partial_x(f(h)) = D\partial_x^2 h - \partial_x^4 h. \tag{6.2}$$

In fact, it is the fourth order diffusion that produces the undercompressive shocks. Numerical simulation of (6.2) shows that similar structures occur in this case. The main difference between (2.1) and (6.2) is that the degeneracy in the diffusion in (2.1), in particular in the fourth order term, causes some singular behavior to occur for very small values of h. Numerical computations of the traveling waves for $D = 0$ show that as $b \to 0$, the value of s for which the undercompressive wave occurs approaches $s = 0$ while the value of t approaches $t = 1$. For very small values of b, jump initial data corresponding to very weak Lax shocks evolve to a solution of (6.1) with two shocks, with the special undercompressive wave as the leading shock. Since this undercompressive wave connects $t \sim 1$ to $b \sim 0$, we obtain a solution that reaches a height of order one from initial data of small order. This is a beautiful example of a violation of the maximum principle for convection-diffusion problems of higher order. An open theoretical problem is to prove that with the nonlinear diffusion in (1.1), the undercompressive waves have such singular limits as $b \to 0$.

Numerical simulations of Münch [BMS99, M99] show that the undercompressive wave is the limit of a cascade of bifurcations that occur as the shock speed varies in (2.11). In particular, for small values of D, the phase portrait of the ODE at the critical speed s_* at which the undercompressive wave occurs has unusual structure. The unstable manifold from T is part of the topological boundary of the (two-dimensional) unstable manifold from M, which wraps around the unstable manifold from T in a spiral with an infinite number of turns. The result is that at the critical speed s_*, not only does the unstable manifold from T connect to B but the unstable manifold from M intersects the stable manifold of B an infinite number of times; there are an infinite number of compressive waves connecting M to B with the undercompressive wave from T to B as their limit. Part of this structure is reminiscent of Silnikov's example [GH86] and we expect that machinery to be useful in studying this problem.

Finally we note that stability of traveling waves yields another interesting set of problems. Numerical simulations show that the undercompressive traveling waves, as solutions of (1.1), are stable with respect to perturbations. However, when there are multiple compressive waves at the same speed and with identical far field states, then some are stable and some are unstable.

A physically relevant problem is to gain more theoretical insight into the stability of traveling waves as plane wave solutions of the two-dimensional PDE

$$\partial_t h + \partial_x(h^2 - h^3) = D\nabla \cdot (h^3 \nabla h) - \nabla \cdot (h^3 \nabla \Delta h). \tag{6.3}$$

This is an important problem for understanding fingering patterns in driven film flow. Numerical simulations (with small D) show that compressive waves are typically unstable to transverse perturbations [THSJ89, BB97, KT97] while undercompressive waves are stable to transverse perturbations [BMFC98, KT98]. These stability differences are reflected in recent and ongoing experiments.

Recent progress has been made in understanding stability of undercompressive waves in systems of conservation laws [GZ98, LZ95]. These techniques will be used to explore stability to one- and two-dimensional perturbations from a more theoretical point of view in the near future [BMSZ99].

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