MA 225 Foundations of Advanced Mathematics Third Test Spring, 2018

Solutions

1. (30 points) Are the following sets empty, finite, denumerable or uncountable? Explain your answer in each case. For each finite set A, state the number of elements $\#A$.

(i) $A = \{x \in \mathbb{Z} : x^2 - 6x - 8 \leq 0.\}$ **Ans.** finite. $\#A = 9$.

(ii) $\{x \in \mathbb{R} : x^2 - 6x + 8 < 0\}$ **Ans.** Uncountable

(iii) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y^2 = 1\}$ **Ans.** Uncountable

(iv) $B = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x + y^2 \leq 3\}$ **Ans.** finite. $\#B = 2$.

2. (50 points) Consider functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} x - 1 & \text{if} \quad x < 1 \\ \dfrac{1}{x} - 1 & \text{if} \quad x \geq 1 \end{cases} \qquad g(x) = \begin{cases} 2x & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad x \geq 0. \end{cases}$$

(a) Write formulae for the functions $f \circ g : \mathbb{R} \longrightarrow \mathbb{R}$ and $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$.

Ans. $$f \circ g(x) = \begin{cases} 2x - 1 & \text{if} \quad x \leq 0 \\ x^2 - 1 & \text{if} \quad 0 < x < 1 \\ \dfrac{1}{x^2} - 1 & \text{if} \quad x \geq 1 \end{cases} \qquad g \circ f(x) = \begin{cases} 2(x - 1) & \text{if} \quad x < 1 \\ \dfrac{2}{x} - 2 & \text{if} \quad x \geq 1 \end{cases}$$

(b) State whether the functions f and g are 1-1 or onto and prove your answers.

Ans. f is neither $1-1$ nor onto. **Proof:** $f(1/2) = f(2) = -1/2$, so not $1-1$; $f(x) \leq 0$ so $y = 1 \notin range f$, therefore, not onto. #
g is $1-1$ and onto. **Proof:** Suppose $g(x) = g(y)$. Then 3 cases: (i) If $x < 0 \leq y$, then $g(x) < 0 \leq g(y)$. This is a contradiction so cannot hold.
(ii) If both $x, y < 0$, then $2x = 2y$, so that $x = y$.
(iii) If both $x, y > 0$, then $x^2 = y^2$, so that $x = y > 0$. #

3. (20 points) Let A, B, C be sets, and let $f : A \longrightarrow B, g : B \longrightarrow C$ be functions.

Prove: If $g \circ f : A \longrightarrow C$ is 1-1, then f is 1-1.
Proof: Suppose $g \circ f$ is $1-1$. Let $f(x) = f(y)$. Then $g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y)$. Hence, $x = y$, since $g \circ f$ is $1-1$. #